International Journal of Theoretical Physics, Vol. 36, No. 11, 1997

Higher Order Gamow States with Exponential Decay

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Received June 25, 1997

We derive Gamow vectors from S-matrix poles of higher multiplicity in analogy to the Gamow vectors describing resonances from first-order poles. With these vectors we construct a density operator that describes resonances associated with higher order poles that obey an exponential decay law. It turns out that this operator formed by these higher order Gamow vectors has a unique structure.

1. INTRODUCTION

Resonances in quantum mechanics can be described by poles of the analytically continued S-matrix on the second sheet of a two-sheeted Riemann surface (Newton, 1982; Goldberger and Watson, 1964a, b; Bohm, 1979, 1980, 1981, 1993). These poles appear in conjugate pairs below (at $z_R = E_R - i\Gamma/2$) and above (at $z_R^* = E_R + i\Gamma/2$) the positive real axis, where the pole at z_R corresponds to a decaying state at times $t \geq 0$ and the pole at z_R^* to the growing state at $t < 0$. These poles lead to the description of the Gamow vectors with energy E_R and lifetime $\tau = \hbar/\Gamma$. The Gamow vectors possess all the properties of resonances, in particular, an exponential decay law and a Breit–Wigner energy distribution. One can extend this derivation of the Gamow vectors from first-order poles to S-matrix poles of higher multiplicity (Antoniou and Gadella, 1995; Bohm *et al.*, 1995, 1997). An S-matrix pole of order r at the position $z_R = E_R - i\Gamma/2$ on the second Riemann sheet leads to a set of r generalized eigenvectors of the Hamiltonian of order $k = 0, 1, \ldots, r - 1$, which are Jordan vectors of degree $k + 1$ to the generalized eigenvalue $E_R - i\Gamma/2$ and which are elements of a generalized complex eigenvector expansion (nuclear spectral theorem in the rigged Hilbert space).

0020-7748/97/1100-2295$12.50/0

The form of this generalized complex eigenvector expansion suggests the definition of a state operator (density matrix) for the microphysical decaying state from a higher order pole. This microphysical state is not a pure state, but a mixture of nonreducible components. In spite of the fact that the kth-order Gamow–Jordan vectors have a polynomial time dependence besides the exponential, which in the past were always associated with resonances from higher order poles, this microphysical state obeys a purely exponential decay law.

Resonances from higher order poles, in particular double poles, were already described about 30 years ago (Goldberger and Watson, 1964a, b; Newton, 1982; Goldhaber, 1968), but were always associated with an additional polynomial time dependence not confirmed in experiment. However, operators containing finite-dimensional matrices consisting of nondiagonalizable Jordan blocks have been discussed in connection with resonances numerous times in the past (Mondragón, 1994; Stodolsky, 1970; Lukierski, 1967; Dothan and Horn, 1970; Katznelson, 1980; Bhamathi and Sudarshan, 1996; Brändas and Chatzidimitriou-Dreismann, 1987; Antoniou and Tasaki, 1993). Using the formalism of the rigged Hilbert space, I. Antoniou and M. Gadella derived the Gamow–Jordan vectors, or higher-order Gamow vectors, from the higher-order poles of the S-matrix.

In Section 2 we recall some of the notation (Bohm, 1993; Bohm *et al.*, 1997) needed for the description of the scattering experiment in the rigged Hilbert space formalism. We show how to obtain these hypothetical vectors associated with the higher order S-matrix poles and show that they are Jordan vectors (Baumgärtel, 1984; Kato, 1966; Gantmacher, 1959). We derive their properties, under the action of the Hamiltonian and their semigroup time evolution. In Section 3 we discuss possible operators formed by these vectors to represent microphysical states describing resonances. In Section 4 we will ask for the converse: Going out from an exponential decay law for the time evolution of resonances from higher-order S-matrix poles, what is the most general form of the operator formed by dyadic products of higher order Gamow vectors?

2. POLES OF THE S-MATRIX AND GAMOW–JORDAN VECTORS

We recall that in the rigged Hilbert space formalism one uses two different space triplets for the set of in-states $\phi^+ \in \Phi_- \subset \mathcal{H} \subset \Phi_-^\times$ describing the preparation process of the scattering experiment and the set of observables $\psi^- \in \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times$ describing the registration process (Bohm *et al.*, 1997; Bohm and Gadella, 1989). The in-state ϕ^+, which evolves from the prepared in-state ϕ^{in} outside the interaction region, is determined by the accelerator.

The so-called out-state ψ^- (or ψ^{out}) is determined by the detector. $|\psi^{\text{out}}\rangle\langle\psi^{\text{out}}|$ is therefore the observable which the detector registers and not a state. The S-matrix elements are given by the projection of the set of out-states $\{\phi^{\text{out}}\}$ onto the set of observables $\{\psi^{\text{out}}\}$ (Bohm, 1993)

$$(\psi^{\text{out}}, \phi^{\text{out}}) = (\psi^{\text{out}}, S\phi^{\text{in}}) = (\psi^-, \phi^+) = \int_{\text{spec}H} dE\, \langle\psi^-|E^-\rangle S(E)\langle^+E|\phi^+\rangle \tag{2.1}$$

The vectors $|E^{\pm}\rangle \in \Phi^{\times}$ are the scattering states (Dirac kets) and are eigenvectors of the exact Hamiltonian with energy label E, which can take values on a two-sheeted Riemann surface. We choose to ignore all other labels of the basis vectors $|E^{\pm}\rangle$, since nothing important is gained in our discussion if we retain the additional quantum numbers, e.g., the angular momentum quantum numbers l and l_3 or the polarization or channel quantum numbers η. Thus, we shall restrict our discussion to one initial channel $\eta = \eta_A$ and one final channel $\eta' = \eta_B$ (e.g., $\eta_B = \eta_A$ for elastic scattering), and we shall consider the lth partial wave of the η_Bth channel (Bohm, 1993): i.e., $S(E) \equiv S_l^{\eta_B}(E)$. We consider the model in which the S-matrix is analytically continued to a two-sheeted Riemann surface in the energy representation (Bohm, 1993) and in which the S-matrix $S(\omega)$, $\omega \in \mathbb{C}$, has one rth-order pole at the position $\omega = z_R(z_R = E_R - i\Gamma/2)$ in the lower half-plane of the second sheet (and consequently there is also one rth-order pole in the upper half-plane of the second sheet at $\omega = z_R^*$). In this paper we will not discuss the pole at z_R^*, as it leads to r growing higher-order Gamow vectors, and the correspondence between the growing and decaying vectors is just the same as for first-order pole resonances ($r = 1$). The model that we discuss here can easily be extended to any finite number of finite-order poles in the second sheet below the positive real axis.

The unitary S-matrix of a quasistationary state associated with an rth-order pole at $z_R = E_R - i\Gamma/2$ in the lower half-plane of the second sheet (denoted by II) is represented by (Bohm, 1993, Section XVIII.6)

$$S_{\text{II}}(\omega) = e^{2i\delta_R(\omega)}e^{2i\gamma(\omega)} \tag{2.2}$$

where $\delta_R(\omega) = r \arctan [\Gamma/2(E_R - \omega)]$ is the rapidly varying resonant part of the phase shift, and $\gamma(\omega)$ is the background phase shift, which is a slowly varying function of the complex energy ω. Here r is a dimensionless quantity that, due to the analyticity properties of the S-matrix (Bohm, 1993, Section XVIII.6), takes integer values, where $r > 0$ leads to a decaying resonance of order r, and $r < 0$ to its corresponding growing state. Using the identity

$$\arctan \frac{\Gamma/2}{E_R - \omega} = \frac{i}{2}\left[\ln\left(\omega - E_R - \frac{i\Gamma}{2}\right) - \ln\left(\omega - E_R + \frac{i\Gamma}{2}\right)\right]$$

one can rewrite $S_{\text{II}}(\omega)$:

$$S_{\text{II}}(\omega) = \left(\frac{\omega - E_R - i\Gamma/2}{\omega - (E_R - i\Gamma/2)}\right)^r e^{2i\gamma(\omega)} = e^{2i\gamma(\omega)} + \sum_{l=1}^{r} \binom{r}{l} \frac{(-i\Gamma)^l}{(\omega - z_R)^l} e^{2i\gamma(\omega)} \tag{2.3}$$

We insert this into (2.1) and deform the contour of integration $\mathscr{C}_-$ through the cut along the spectrum of H into the second sheet (Bohm, 1979, 1980, 1981, 1993). Then one obtains

$$\begin{aligned}(\psi^-, \phi^+) = &\int_{\mathscr{C}_-} d\omega\, \langle\psi^-|\omega^-\rangle S_{\text{II}}(\omega)\langle^+\omega|\phi^+\rangle \\ &+ \sum_{n=0}^{r-1} \binom{r}{n+1} (-i\Gamma)^{n+1} \\ &\times \oint_{\hookleftarrow} d\omega\, \langle\psi^-|\omega^-\rangle \frac{e^{2i\gamma(\omega)}}{(\omega - z_R)^{n+1}} \langle^+\omega|\phi^+\rangle, \qquad \operatorname{Im}(\omega) < 0\end{aligned} \tag{2.4}$$

The first integral does not depend on the pole and is called the "background term." The contour $\mathscr{C}_-$ can be deformed into the negative axis of the second sheet from 0 to $-\infty_{\text{II}}$:

$$(\psi^-, \phi^+) = \int_0^{-\infty_{\text{II}}} dE\, \langle\psi^-|E^-\rangle S_{\text{II}}(E)\langle^+E|\phi^+\rangle + (\psi^-, \phi^+)_{\text{P.T.}} \tag{2.5}$$

We will not need to further investigate the background integral in this presentation. For the higher order pole term $(\psi^-, \phi^+)_{\text{P.T.}}$ we obtain, using the Cauchy integral formula

$$\oint_{\hookleftarrow} \frac{f(\omega)}{(\omega - z_R)^{n+1}}\, d\omega = \frac{2\pi i}{n!} f^{(n)}(z)\Big|_{z=z_R}$$

where $f^{(n)}(z) \equiv d^n f(z)/dz^n$:

$$(\psi^-, \phi^+)_{\text{P.T.}} = \sum_{n=0}^{r-1} \binom{r}{n+1} (-i\Gamma)^{n+1} \left(-\frac{2\pi i}{n!}\right) \left(\langle\psi^-|\omega^-\rangle e^{2i\gamma(\omega)} \langle^+\omega|\phi^+\rangle\right)^{(n)}_{\omega=z_R} \tag{2.6}$$

where $(\cdots)^{(n)}_{\omega=z_R}$ denotes the nth derivative with respect to ω taken at the value $\omega = z_R$.

Since the kets $|\omega^-\rangle$ are (like the Dirac kets $|E^-\rangle$) only defined up to an arbitrary factor or, if their "normalization" is already fixed, up to a phase factor, we can absorb the background phase $e^{2i\gamma(\omega)}$ into the kets $|\omega^-\rangle$ and define

$$|\omega^\gamma\rangle \equiv |\omega^-\rangle e^{2i\gamma(\omega)} \tag{2.7}$$

Note that this phase is not trivial, e.g., $|E^+\rangle = |E^-\rangle S_{\mathrm{II}}(E) = |E^-\rangle e^{2i\delta_R(E)} e^{2i\gamma(E)}$, except for the case when the slowly varying background phase $\gamma(\omega)$ is constant and the $|\omega^\gamma\rangle$ are identical with $|\omega^-\rangle$ up to a totally trivial constant phase factor. In general, (2.7) is a nontrivial gauge transformation. We will keep the phase γ in our Gamow vectors, but not investigate their properties further. The results of this section are true also for the case when $|z_R^\gamma\rangle^{(k)}$ are exchanged by $|z_R^-\rangle^{(k)}$. But as we cannot just ignore the existence of the background integral in (2.4), we have to keep in mind that their existence is not irrelevant, if one deals with poles of order $r > 1$.

Taking the derivatives, we write the pole term as

$$(\psi^-, \phi^+)_{\mathrm{P.T.}} = \sum_{n=0}^{r-1} \binom{r}{n+1} (-i\Gamma)^{n+1} \left(-\frac{2\pi i}{n!}\right) \times \sum_{k=0}^{n} \binom{n}{k} \langle \psi^- | z_R^\gamma\rangle^{(k)}\, {}^{(n-k)}\langle {}^+z_R | \phi^+\rangle \tag{2.8}$$

where we denote the nth derivative of the analytic function $\langle\psi^-|z^\gamma\rangle$ by $\langle\psi^-|z^\gamma\rangle^{(n)}$ with value $\langle\psi^-|z_R^\gamma\rangle^{(n)}$ at $z = z_R$. Since $\langle\psi^-|E^-\rangle \in \mathcal{S} \cap \mathcal{H}_-^2$, i.e., element of the Schwartz space and of Hardy class (Duren, 1970; Hoffman, 1962; Bohm and Gadella, 1989), it follows that $\langle\psi^-|z^\gamma\rangle^{(n)}$ is also an analytic function in the lower half-plane of the second sheet, whose boundary value on the positive real axis $\langle\psi^-|E^\gamma\rangle^{(n)} \in \mathcal{S} \cap \mathcal{H}_-^2$. Analogously, we denote by ${}^{(n)}\langle{}^+z|\phi^+\rangle$ the nth derivative of the analytic function $\langle{}^+z|\phi^+\rangle$. Again, ${}^{(n)}\langle{}^+z|\phi^+\rangle$ is analytic in the lower half-plane with its boundary value on the real axis being ${}^{(n)}\langle{}^+E|\phi^+\rangle \in \mathcal{S} \cap \mathcal{H}_-^2$. The rth-order pole is therefore associated with the set of r generalized vectors

$$|z_R^\gamma\rangle^{(0)}, \quad |z_R^\gamma\rangle^{(1)}, \ldots, \quad |z_R^\gamma\rangle^{(k)}, \ldots, \quad |z_R^\gamma\rangle^{(n)} \tag{2.9}$$

For the first-order resonance pole this, of course, reduces to the single vector $|z_R^\gamma\rangle = |z_R^\gamma\rangle^{(0)}$ in agreement with Bohm (1979, 1980, 1981, 1993).

We can now establish the complex basis vector expansion in analogy to the Dirac basis vector expansion [nuclear spectral theorem in the rigged Hilbert space (Gel'fand and Vilenkin, 1964; Bohm and Gadella, 1989)]. If we return to the complete S-matrix element (2.4) and insert the pole term, we get

$$(\psi^-, \phi^+) = \int_0^{-\infty_{\mathrm{II}}} dE\, \langle\psi^-|E^+\rangle\langle{}^+E|\phi^+\rangle - \sum_{n=0}^{r-1} \binom{r}{n+1} \frac{2\pi\Gamma}{n!} (-i\Gamma)^n \sum_{k=0}^{n} \binom{n}{k} \langle\psi^-|z_R^\gamma\rangle^{(k)}\, {}^{(n-k)}\langle{}^+z_R|\phi^+\rangle \tag{2.10}$$

Omitting the arbitrary $\psi^- \in \Phi_+$ and rearranging the sums in the second term, we obtain the complex basis vector expansion for an arbitrary $\phi^+ \in \Phi_-$,

$$\phi^+ = \int_0^{-\infty_{II}} dE|E^+\rangle\langle^+E|\phi^+\rangle + \sum_{k=0}^{r-1} b_k|z_R^\gamma\rangle^{(k)} \tag{2.11}$$

where the coefficients b_k are given by

$$b_k = (-2\pi\Gamma) \sum_{n=k}^{r-1} \binom{r}{n+1} \binom{n}{k} \frac{(-i\Gamma)^n}{n!} {}^{(n-k)}\langle^+z_R|\phi^+\rangle \tag{2.12}$$

This complex generalized basis vector expansion is the most important result of our irreversible quantum theory (as is the Dirac basis vector expansion for reversible quantum mechanics). It shows that the generalized vectors (2.9) (functionals over the space Φ_+) are part of a basis system for the $\phi^+ \in \Phi_-$ and form together with the kets $|E^+\rangle$, $-\infty_{II} < E \leq 0$, a complete basis system. The vectors (2.9) span a linear subspace $\mathcal{M}_{z_R} \subset \Phi_+^\times$ of dimension r:

$$\mathcal{M}_{z_R} = \left\{ \xi \middle| \xi = \sum_{k=0}^{r-1} \zeta_k|z_R^\gamma\rangle^{(k)};\ \zeta_k \in \mathbb{C} \right\} \subset \Phi_+^\times \tag{2.13}$$

If there are N poles at z_{R_i} of order r_i, then for every pole there exists a linear subspace $\mathcal{M}_{z_{R_i}} \subset \Phi_+^\times$.

Note that the label k of the higher order Gamow vectors is not a quantum number in the usual sense. Basis vectors are usually labeled by quantum numbers associated with eigenvalues of a complete system of commuting observables (Bohm, 1993, Chapter IV). But there is no physical observable to which the label k is connected. Therefore, the different $|z_R^\gamma\rangle^{(k)}$ in the subspace $\mathcal{M}_{z_R}$ do not have a separate physical meaning.

Now that (2.11) has established the generalized vectors (2.9) as members of a basis system (together with the $|E^+\rangle$; $-\infty_{II} < E \leq 0$) in $\Phi_+^\times$, we can obtain the action of the Hamiltonian H by the action of the operator $H^\times$ on these basis vectors. We will write this Hamiltonian in terms of its matrix elements in this basis. For this purpose we replace the arbitrary $\psi^- \in \Phi_+$ in (2.10) by $\tilde{\psi}^- = H\psi^-$, which is again an element of Φ_+, and we find (Antoniou and Gadella, 1995; Bohm *et al.*, 1997)

$$\begin{aligned}
\langle\tilde{\psi}^-|z_R^\gamma\rangle^{(k)} &\equiv \langle H\psi^-|z_R^\gamma\rangle^{(k)} \equiv \langle\psi^-|H^\times|z_R^\gamma\rangle^{(k)} \\
&= z_R\langle\psi^-|z_R^\gamma\rangle^{(k)} + \langle\psi^-|z_R^\gamma\rangle^{(k-1)}, \qquad k = 1, \ldots, r-1 \\
\langle\tilde{\psi}^-|z_R^\gamma\rangle^{(0)} &\equiv \langle H\psi^-|z_R^\gamma\rangle^{(0)} = \langle\psi^-|H^\times|z_R^\gamma\rangle^{(0)} = z_R\langle\psi^-|z_R^\gamma\rangle^{(0)}
\end{aligned} \tag{2.14}$$

This can also be written as a set of functional equations over Φ_+:

$$H^\times|z_R^\gamma\rangle^{(k)} = z_R|z_R^\gamma\rangle^{(k)} + k|z_R^\gamma\rangle^{(k-1)}, \qquad k = 1, \ldots, r-1$$
$$H^\times|z_R^\gamma\rangle = z_R|z_R^\gamma\rangle \tag{2.15}$$

This means that $H^\times$ restricted to the subspace $\mathcal{M}_{z_R}$ is a Jordan operator of degree r, and the vectors $|z_R^\gamma\rangle^{(k)}$, $k = 0, 1, 2, \ldots, r-1$, are Jordan vectors of degree $k+1$ (Baumgärtel, 1984; Kato, 1966; Gantmacher, 1959). They fulfill the generalized eigenvector equation (Lancaster and Tismenetsky, 1985; see also . . .)

$$(H^\times - z_R)^{k+1}|z_R^\gamma\rangle^{(k)} = 0 \tag{2.16}$$

Since, according to (2.11), the basis system also includes the $|E^+\rangle$, $-\infty_{\text{II}} < E \leq 0$, we indicate this by a continuously infinite diagonal matrix equation

$$(\langle H\psi^-|E^+\rangle) = (\langle\psi^-|H|E^+\rangle) = (E)(\langle\psi^-|E^+\rangle) \tag{2.17}$$

where $(\langle\psi^-|E^+\rangle)$ indicates a continuously infinite column matrix. Then (2.14) can be rewritten as

$$\begin{pmatrix} \langle\psi^-|H^\times|z_R^\gamma\rangle^{(0)} \\ \langle\psi^-|H^\times|z_R^\gamma\rangle^{(1)} \\ \vdots \\ \vdots \\ \langle\psi^-|H^\times|z_R^\gamma\rangle^{(r-1)} \\ \langle\psi^-|H^\times|E^+\rangle \end{pmatrix}$$

$$= \begin{pmatrix} z_R & 0 & 0 & \cdots & 0 & 0 \\ 1 & z_R & 0 & \cdots & 0 & \\ 0 & 2 & z_R & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \\ 0 & 0 & \cdots & r-1 & z_R & 0 \\ 0 & & \cdots & & 0 & (E) \end{pmatrix} \begin{pmatrix} \langle\psi^-|z_R^\gamma\rangle^{(0)} \\ \langle\psi^-|z_R^\gamma\rangle^{(1)} \\ \vdots \\ \vdots \\ \langle\psi^-|z_R^\gamma\rangle^{(r-1)} \\ \langle\psi^-|E^+\rangle \end{pmatrix} \tag{2.18}$$

In this matrix representation of $H^\times$, the upper left $r \times r$ submatrix associated with the complex eigenvalue z_R is a (lower) Jordan block of degree r. One can attain the standard form of a Jordan block with 1's on the lower diagonal by simply choosing the normalization

$$|z_R^\gamma\rangle^{(k)} \to \frac{1}{k!}|z_R^\gamma\rangle^{(k)} \qquad \text{and} \qquad {}^{(l)}\langle{}^+z_R| \to {}^{(l)}\langle{}^+z_R|\frac{1}{l!} \tag{2.19}$$

Next we discuss the time evolution of the higher order Gamow vectors. We replace the arbitrary $\psi^- \in \Phi_+$ in (2.10) by $\tilde{\psi}^- = e^{iHt}\psi^-$. We recall that e^{iHt} needs to be a continuous operator with respect to the topology τ_{Φ_+} of the space Φ_+, and its values e^{izt} need to be holomorphic in all Φ_+. Its conjugate $(e^{iHt})^\times$, which acts on the vectors $|\omega^-\rangle \in \Phi_+$, is only defined for positive values of the parameter t (semigroup time evolution),

$$\langle e^{iHt}\psi^-|\omega^-\rangle = \langle\psi^-|(e^{iHt})^\times|\omega^-\rangle = e^{-i\omega t}\langle\psi^-|\omega^-\rangle, \qquad t \geq 0 \tag{2.20}$$

for all $|\omega^-\rangle \in \Phi_+^\times$. Then

$$\begin{aligned}\langle e^{iHt}\psi^-|z_R^\gamma\rangle^{(k)} &= \frac{d^k}{d\omega^k}\left(\langle e^{iHt}\psi^-|\omega^-\rangle e^{2i\gamma(\omega)}\right)_{\omega=z_R} \\ &= \frac{d^k}{d\omega^k}\left(e^{-i\omega t}\langle\psi^-|\omega^-\rangle e^{2i\gamma(\omega)}\right)_{\omega=z_R} \\ &= e^{-iz_R t}\sum_{p=0}^{k}\binom{k}{p}(-it)^p\langle\psi^-|z_R^\gamma\rangle^{(k-p)}\end{aligned} \tag{2.21}$$

This can be written as a functional equation as

$$(e^{iHt})^\times|z_R^\gamma\rangle^{(k)} = e^{-iz_R t}\sum_{p=0}^{k}\binom{k}{p}(-it)^{k-p}|z_R^\gamma\rangle^{(p)} \tag{2.22}$$

In the same way one derives for the complex conjugate

$$^{(l)}\langle^\gamma z_R|e^{iHt} = e^{iz_R^* t}\sum_{q=0}^{l}\binom{l}{q}(it)^{l-q}\,{}^{(q)}\langle^\gamma z_R| \tag{2.23}$$

The same formulas apply also to the vectors $|z_R^-\rangle^{(k)}$ and $^{(l)}\langle^- z_R|$ with background phase $\gamma = 0$. It is important to note that the time evolution operator $(e^{iHt})^\times$ transforms between different $|z_R^\gamma\rangle^{(k)}$, or different $|z_R^-\rangle^{(k)}$, $k = 0, 1, \ldots, n$, that belong to the same pole of order r at $z = z_R$, but the time evolution does not transform out of $\mathcal{M}_{z_R}$.

3. STATES FROM HIGHER ORDER GAMOW VECTORS

Gamow states of zeroth order with their empirically well-established properties (exponential time evolution, Breit–Wigner energy distribution) have been abundantly observed in nature as resonances and decaying states. Theoretically, there should be no reason why quasistationary states [i.e., states that also cause large time delay in a scattering process (Bohm, 1993, Chapter 18)] associated with integers $r > 1$ in (2.3) should not exist. However, no such quasistationary states have so far been established empirically. One argument against their existence was that the polynomial time dependence, which was always vaguely associated with higher order poles (Goldberger and Watson, 1964a, b; Newton, 1982; Goldhaber, 1968), has not been observed for quasistationary states. The question that we want to discuss in this section is whether there is an analogous physical interpretation for the higher order Gamow vectors as for the ordinary Gamow vectors, namely as states which decay (for $t > 0$) or grow (for $t < 0$) in one preferred direction of time ("arrow of time") and obey the exponential law.

In analogy to von Neumann's definition of a pure stationary state using dyadic products $|f\rangle\langle f|$ of the energy eigenvectors $|f\rangle$ in Hilbert space, microphysical Gamow states connected with first-order poles can be defined as dyadic products of zeroth-order Gamow vectors (Bohm, 1979, 1980, 1981, 1993; Bohm *et al.*, 1997),

$$W^{(0)} = |z_R^-\rangle\langle^- z_R| \tag{3.1}$$

[Note that this definition is not related to the scattering background phase γ which entered in (2.7).] The time evolution of this Gamow state is exponential,

$$\begin{aligned} W^G(t) &\equiv (e^{iHt})^\times |z_R^-\rangle\langle^- z_R| e^{iHt} \\ &= e^{-iz_Rt}|z_R^-\rangle\langle^- z_R| e^{iz_R^*t} \\ &= e^{-i(E_R - i(\Gamma/2))t}|z_R^-\rangle\langle^- z_R| e^{i(E_R + i(\Gamma/2))t} \\ &= e^{-\Gamma t} W^G(0), \qquad t \geq 0 \end{aligned} \tag{3.2}$$

Mathematically, equation (3.2) is understood as the functional equation of

$$\langle\psi^-|W^G(t)|\psi^-\rangle = e^{-\Gamma t}\langle\psi^-|W^G|\psi^-\rangle \qquad \text{for all} \quad \psi^- \in \Phi_+ \quad \text{and} \quad t \geq 0$$

This shows how important it is in the RHS formalism of quantum mechanics to know what question one wants to ask about a microphysical state when one makes the hypothesis (3.1). The vectors $\psi^- \in \Phi_+$ represent observables defined by the detector (registration apparatus), and therefore the operator W^G represents the microsystem that affects the detector. Therefore the quantity $\langle\psi^-|W^G|\psi^-\rangle$ is the answer to the question, *What is the probability that the microsystem affects the detector*?

If the detector is triggered at a later time t, i.e., when the observable has been time translated

$$|\psi^-\rangle\langle\psi^-| \to e^{iHt}|\psi^-\rangle\langle\psi^-|e^{-iHt} = |\psi^-(t)\rangle\langle\psi^-(t)| \tag{3.3}$$

then the same question for $t \geq 0$ has the following answer: The probability that the microsystem affects the detector at $t > 0$ is

$$\begin{aligned}\langle\psi^-(t)|W^G|\psi^-(t)\rangle &= \langle e^{iHt}\psi^-|W^G|e^{iHt}\psi^-\rangle \\ &= \langle\psi^-|(e^{iHt})^\times W^G e^{iHt}|\psi^-\rangle \\ &= e^{-\Gamma t}\langle\psi^-|W^G|\psi^-\rangle \end{aligned} \tag{3.4}$$

This means that (3.4) is the probability to observe the decaying microstate at a time t relative to the probability $\langle\psi^-|W^G|\psi^-\rangle$ at $t = 0$ [which one can "normalize" to unity by choosing the appropriate factor on the right-hand side of (3.1)].

The question that one asks in the scattering experiment is different. There the pole term (P.T.) of (2.5) for $r = 1$ describes how the microsystem propagates the effect which the preparation apparatus (accelerator, described by the state ϕ^+) causes on the registration apparatus (detector, described by the observable ψ^-). This involves both the observables $\psi^- \in \Phi_+$ and the prepared states $\phi^+ \in \Phi_-$, and one would ask the question: *What is the probability to observe $\psi^-(t)$ in a microphysical resonance state of a scattering experiment with the prepared in-state ϕ^+?*

In distinction to the decay experiment, where one just asks for the probability of $\psi^- \in \Phi_+$, in the resonance scattering experiment one asks for the probability that relates $\psi^- \in \Phi_+$ to $\phi^+ \in \Phi_-$ via the microphysical resonance state. Therefore the mathematical quantity that describes the microphysical resonance state in a scattering experiment cannot be given by $|z_R^-\rangle\langle^- z_R|$, but must be given by something like $|z_R^\gamma\rangle\langle^+ z_R|$. The probability to observe ψ^- in the prepared state ϕ^+, independently of how the effect of ϕ^+ is carried to the detector ψ^-, is given by the S-matrix element (2.1), $|(\psi^-, \phi^+)|^2$. The probability amplitude that this effect is carried by the microphysical resonance state is then given by its pole term $(\psi^-, \phi^+)_{\text{P.T.}}$. In analogy to the decay experiment, one can also compare these probabilities at different times. For this purpose one translates the observable ψ^- in time by an amount $t \geq 0$,

$$\psi^- \to \psi^-(t) = e^{iHt}\psi^-; \qquad t \geq 0 \tag{3.5}$$

Physically, this would correspond to turning on the detector at a time $t \geq 0$ later than for ψ^-. One obtains

$$\begin{aligned}(\psi^-(t), \phi^+)_{\text{P.T.}} &= -2\pi\Gamma\langle e^{iHt}\psi^-|z_R^\gamma\rangle\langle^+ z_R|\phi^+\rangle \\ &= -2\pi\Gamma e^{-iz_R t}\langle\psi^-|z_R^\gamma\rangle\langle^+ z_R|\phi^+\rangle \\ &= e^{-iE_R t}e^{-\Gamma t/2}(\psi^-, \phi^+)_{\text{P.T.}}\end{aligned} \tag{3.6}$$

This means that the time-dependent probability, due to the first-order pole term, to measure the observable $\psi^-(t)$ in the state ϕ^+ is given by the exponential law

$$|(e^{iHt}\psi^-, \phi^+)_{\text{P.T.}}|^2 = e^{-\Gamma t}|(\psi^-, \phi^+)_{\text{P.T.}}|^2 \tag{3.7}$$

This is as one would expect if the action of the preparation apparatus on the registration apparatus is carried by an exponentially decaying microsystem (resonance) described by a Gamow vector.

Thus we have seen that there are two ways in which a resonance associated with a first-order pole of the S-matrix ($r = 1$) can appear in experiments, and therefore there are two different forms to represent the decaying Gamow state:

$$\text{in a decay experiment:} \qquad |z_R^-\rangle\langle^- z_R| \tag{3.8a}$$

$$\text{in a scattering experiment:} \qquad |z_R^\gamma\rangle\langle^+ z_R| \tag{3.8b}$$

An analogous statement holds for the Gamow states associated with the pole in the upper half-plane. The first representation is the one used in the S-matrix when one calculates the cross section; the second representation is the one used when one calculates the Golden Rule (decay rate). In contrast to von Neumann's formulation, where a given state (representing an ensemble prepared by the preparation apparatus) is always described by one and the same density operator $|f\rangle\langle f|$, the representation of the microphysical state in the RHS formulation depends upon the kind of experiment one performs. That a theory of the microsystems must include the methods of the experiments has previously been emphasized by G. Ludwig.

We will now discuss the mathematical representations of Gamow states associated with higher order poles of the S-matrix ($r > 1$) under the two aspects above. In analogy to the case for $r = 1$, we conjecture that the representation for the microphysical system in the resonance scattering experiment is already determined by the pole term (2.8), and is therefore given by

$$\begin{aligned} W_{\text{P.T.}} &= 2\pi\Gamma \sum_{n=0}^{r-1} \binom{r}{n+1} (-i)^n \frac{\Gamma^n}{n!} \sum_{k=0}^{n} \binom{n}{k} |z_R^\gamma\rangle^{(k)}\,{}^{(n-k)}\langle^+ z_R| \\ &= 2\pi\Gamma \sum_{n=0}^{r-1} \binom{r}{n+1} (-i)^n W_{\text{P.T.}}^{(n)} \end{aligned} \tag{3.9}$$

up to a normalization factor which will have to be determined by normalizing the overall probability to 1. Here we define the operator

$$W_{\text{P.T.}}^{(n)} = \frac{\Gamma^n}{n!} \sum_{k=0}^{n} \binom{n}{k} |z_R^\gamma\rangle^{(k)}\,{}^{(n-k)}\langle^+ z_R| \tag{3.10}$$

We hypothesize that the microphysical state from higher order poles connected with the decay experiment has the same structure as the microphysical state

(3.9), which is certainly in agreement with the first-order case (3.8a) in comparison with (3.8b),

$$W = 2\pi\Gamma \sum_{n=0}^{r-1} \binom{r}{n+1} (-i)^n \frac{\Gamma^n}{n!} \sum_{k=0}^{n} \binom{n}{k} |z_R^-\rangle^{(k)\ (n-k)}\langle^- z_R|$$

$$= 2\pi\Gamma \sum_{n=0}^{r-1} \binom{r}{n+1} (-i)^n W^{(n)} \tag{3.11}$$

For $r = 1$, this reduces to (3.8a). It should be mentioned that mathematically there is an important difference between (3.9) and (3.11) because the $\langle\psi^-|z^\gamma\rangle^{(k)\ (n-k)}\langle^+ z|\phi^+\rangle$ are analytic functions for z in the lower half-plane, whereas the $\langle\psi_1^-|z^-\rangle^{(k)\ (n-k)}\langle^- z|\psi_2^-\rangle$ are not. Whether the microphysical state of the (hypothetical) quasistationary microphysical system is always represented by the mathematical object (3.11) or whether also each individual

$$W^{(n)} = \frac{\Gamma^n}{n!} \sum_{k=0}^{n} \binom{n}{k} |z_R^-\rangle^{(k)\ (n-k)}\langle^- z_R|, \qquad n = 0, 1, \ldots, r-1 \tag{3.12}$$

has a separate physical meaning cannot be said at this point.

This means that the conjectural physical state associated with the rth-order pole is a mixed state W, all of whose components $W^{(n)}$, except for the zeroth component $W^{(0)}$, cannot be reduced further into "pure" states given by dyadic products like $|z_R^-\rangle^{(k)\ (k)}\langle^- z_R|$. This is quite consistent with our earlier remark that the label k is not a quantum number connected with an observable (like the suppressed labels $b_2, \ldots, b_n$). Therefore a "pure state" with a definite value of k, like $|z_R^-\rangle^{(k)\ (k)}\langle^- z_R|$, $k \geq 1$, does not make sense physically. A physical interpretation could only be given to the whole r-dimensional space $\mathcal{M}_{z_R}$, (2.13). The individual $W^{(n)}$, $n = 0, 1, 2, \ldots, r-1$, act in the subspaces $\mathcal{M}_{z_R}^{(n)} \subset \mathcal{M}_{z_R}$ which are spanned by Gamow vectors of order $0, 1, \ldots, n$ [Jordan vectors of degree $n + 1$, i.e., $(H^\times - z_R)^{n+1}\, \mathcal{M}_{z_R}^{(n)} = 0$]. Here the question is whether there could be a physical meaning to each $W^{(n)}$ separately, or whether only the particular mixture W given by (3.11) can occur physically.

Though the quantities $|z_R^-\rangle^{(k)\ (k)}\langle^- z_R|$ will have no physical meaning, even if higher order poles exist, they have been considered in the literature (Antoniou and Gadella, 1995; Bohm *et al.*, 1985); Goldberger and Watson; Newton, 1982; Goldhaber, 1968), and their time evolution is calculated in a straightforward way:

$$(e^{iHt})^\times |z_R^-\rangle^{(k)\ (k)}\langle^- z_R| e^{iHt}$$

$$= e^{-\Gamma t} \sum_{l=0}^{k} \sum_{m=0}^{k} \binom{k}{l} \binom{k}{m} (-it)^l (it)^m |z_R^-\rangle^{(k-l)\ (k-m)}\langle^- z_R| \tag{3.13}$$

This state operator shows the additional polynomial time dependence that has always been considered an obstacle to the use of higher order poles for quasistationary states. A polynomial time dependence should have shown up in many experiments.

We now derive the time evolution of the microphysical state operators (3.11) from higher order poles of the S-matrix using the time evolution obtained for the Gamow–Jordan vector. It will turn out that the operator (3.12) and therewith (3.11) have a purely exponential time evolution. Inserting (2.22) and (2.23) into

$$W^{(n)}(t) = (e^{iHt})^{\times} W^{(n)} e^{iHt} = \sum_{k=0}^{n} \binom{n}{k} (e^{iHt})^{\times} |z_R^-\rangle^{(k)} \, {}^{(n-k)}\langle{}^{-}z_R| e^{iHt}, \qquad t \geq 0 \tag{3.14}$$

we get

$$W^{(n)}(t) = e^{iz_R t} e^{iz_R^* t} \frac{\Gamma^n}{n!} \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{m=0}^{n-k} \binom{n}{k} \binom{k}{l} \binom{n-k}{m} \times (-it)^{k-l} (it)^{n-k-m} |z_R^-\rangle^{(l)} \, {}^{(m)}\langle{}^{-}z_R|$$

After reordering the summations and the terms in the binomial coefficients, one can separate out the dyads,

$$W^{(n)}(t) = e^{-\Gamma t} \frac{\Gamma^n}{n!} \sum_{m=0}^{n} \binom{n}{m} \sum_{l=0}^{n-m} \binom{n-m}{l} |z_R^-\rangle^{(l)} \, {}^{(m)}\langle{}^{-}z_R| \sum_{k=l}^{n-m} \binom{n-m-l}{k-l} (-it)^{k-l} (it)^{n-k-m}$$

Since the indices labeling the Gamow–Jordan vectors do not depend upon k, the sum over k may be performed using the binomial formula

$$\sum_{k=l}^{n-m} \binom{n-m-l}{k-l} (-it)^{k-l} (it)^{n-k-m} = (it - it)^{n-m-l} = \begin{Bmatrix} 1 & \text{for} & l = n-m \\ 0 & \text{for} & l \neq n-m \end{Bmatrix} = \delta_{l,n-m}$$

and one gets

$$W^{(n)}(t) = e^{-\Gamma t} \frac{\Gamma^n}{n!} \sum_{m=0}^{n} \binom{n}{m} |z_R^-\rangle^{(n-m)} \, {}^{(m)}\langle{}^{-}z_R| = e^{-\Gamma t} W^{(n)}(0), \qquad t \geq 0 \tag{3.15}$$

This means that the nonreducible (i.e., "mixed") state operator $W^{(n)}$ of (3.12) has a simple exponential semigroup time evolution, and also that, since the operator W of (3.11) a linear combination of the $W^{(n)}$,

$$W(t) \equiv (e^{iHt})^{\times} W e^{iHt} = e^{-\Gamma t} W, \qquad t \geq 0 \tag{3.16}$$

It turns out that the operator (3.12) is the only operator in $\mathcal{M}_{z_R}^{(n)}$ formed by the dyadic products $|z_R^-\rangle^{(m)}\ {}^{(l)}\langle^- z_R|$ with $m,\ l = 0,\ 1,\ \ldots,\ n$, which has a purely exponential time evolution, thus being distinguished from all other operators in $\mathcal{M}_{z_R}^{(n)}$. Thus we have seen that the state operator which we conjecture from the rth-order pole term describes a nonreducible "mixed" microphysical decaying state which obeys an exact exponential decay law.

In analogy to (3.6), one can also calculate the time evolution of the operators (3.9) and (3.10), but their time evolution will always have an additional polynomial time dependence besides the exponential.

4. GENERAL FORM OF THE EXPONENTIALLY DECAYING GAMOW STATE

In this section we discuss the converse of the above reasoning, where we conjectured the density operator for higher order decaying Gamow states and derived a purely exponential time evolution. We ask the question: If we require exponentially decaying time evolution for a Gamow state operator formed by dyadic products of vectors in $\mathcal{M}_{z_R}$, what is the most general form of such an operator?

We denote the most general form of W of finite dimension $j \in \mathbb{N}$ by

$$W_{(j)}^{\square}(0) = \sum_{k=0}^{j} \sum_{h=0}^{j} A_{hk} |z_R^-\rangle^{(k)}\ {}^{(h)}\langle^- z_R| \tag{4.1}$$

with arbitrary coefficients A_{hk}. We want to change the order of summation from the states of the form $|z_R^-\rangle^{(k)}\ {}^{(h)}\langle^- z_R|$ to the states of the form $|z_R^-\rangle^{(k)}\ {}^{(n-k)}\langle^- z_R|$. Changing the label $h = n - k$, we write this sum as

$$W_{(j)}^{\square}(0) = \sum_{k=0}^{j} \sum_{n=k}^{j+k} A_{n-k,k} |z_R^-\rangle^{(k)}\ {}^{(n-k)}\langle^- z_R| \tag{4.2}$$

Switching the order of k and n divides (4.2) into two sums,

$$\sum_{k=0}^{j} \sum_{n=k}^{j+k} = \sum_{n=0}^{j} \sum_{k=0}^{n} + \sum_{n=j+1}^{2j} \sum_{k=n-j}^{j} \tag{4.3}$$

We define $W_{(j)}^{\square} = W_{(j)}^{\triangledown} + W_{(j)}^{\triangle}$ such that

$$W^{\nabla}_{(j)} = \sum_{n=0}^{j} \sum_{k=0}^{n} A_{n-k,k} |z_R^-\rangle^{(k)\ (n-k)}\langle^- z_R| \tag{4.4}$$

$$W^{\Delta}_{(j)} = \sum_{n=j+1}^{2j} \sum_{k=n-j}^{j} A_{n-k,k} |z_R^-\rangle^{(k)\ (n-k)}\langle^- z_R| \tag{4.5}$$

In the following, we calculate the coefficients of $W^{\nabla}_{(j)}$ and give an argument why this suffices to conclude that the coefficients of $W^{\Delta}_{(j)}$ all turn out to be zero.

The time dependence of W^{∇} is given, using (2.22) and (2.23), by

$$\begin{aligned} W^{\nabla}_{(j)}(t) &= (e^{iHt})^{\times} W^{\nabla}_{(j)} e^{iHt} \\ &= \sum_{n=0}^{j} \sum_{k=0}^{n} A_{n-k,k} (e^{iHt})^{\times} |z_R^-\rangle^{(k)\ (n-k)}\langle^- z_R| e^{iHt} \\ &= e^{-\Gamma t} \sum_{n=0}^{j} \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{m=0}^{n-k} A_{n-k,k} \binom{k}{l} \binom{n-k}{m} \\ &\quad (-it)^{k-l} (it)^{n-k-m} |z_R^-\rangle^{(l)\ (m)}\langle^- z_R| \end{aligned}$$

Changing the order of the summations,

$$\begin{aligned} \sum_{n=0}^{j} \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{m=0}^{n-k} &= \sum_{n=0}^{j} \sum_{l=0}^{n} \sum_{k=l}^{n} \sum_{m=0}^{n-k} \\ &= \sum_{l=0}^{j} \sum_{n=l}^{j} \sum_{k=l}^{n} \sum_{m=0}^{n-k} = \sum_{l=0}^{j} \sum_{n=l}^{j} \sum_{m=0}^{n-l} \sum_{k=l}^{n-m} \\ &= \sum_{l=0}^{j} \sum_{m=0}^{j-l} \sum_{n=l+m}^{j} \sum_{k=l}^{n-m} \end{aligned}$$

allows the dyadic products, which are linearly independent operators, to be factored out of the sums over terms in which they appear as common factors:

$$\begin{aligned} W^{\nabla}_{(j)}(t) &= e^{-\Gamma t} \sum_{l=0}^{j} \sum_{m=0}^{j-l} \sum_{n=l+m}^{j} \sum_{k=l}^{n-m} A_{n-k,k} \binom{k}{l} \binom{n-k}{m} \\ &\quad \times (-it)^{k-l} (it)^{n-k-m} |z_R^-\rangle^{(l)\ (m)}\langle^- z_R| \\ &= e^{-\Gamma t} \sum_{l=0}^{j} \sum_{m=0}^{j-l} |z_R^-\rangle^{(l)\ (m)}\langle^- z_R| \\ &\quad \times \sum_{n=l+m}^{j} \sum_{k=l}^{n-m} A_{n-k,k} \binom{k}{l} \binom{n-k}{m} (-it)^{k-l} (it)^{n-k-m} \\ &= e^{-\Gamma t} \sum_{l=0}^{j} \sum_{m=0}^{j-l} |z_R^-\rangle^{(l)\ (m)}\langle^- z_R| \sum_{n=l+m}^{j} (it)^{n-m-l} \\ &\quad \times \sum_{k=l}^{n-m} A_{n-k,k} \binom{k}{l} \binom{n-k}{m} (-1)^{k-l} \end{aligned}$$

The operator $W^{\nabla}(t)$ will decay according to the pure exponential $e^{-\Gamma t}$ if and only if all terms involving additional powers of t cancel. All terms involving additional powers of t will cancel if and only if the coefficients $A_{n-k,k}$ satisfy the conditions

$$0 = \sum_{k=l}^{n-m} A_{n-k,k} \binom{k}{l} \binom{n-k}{m} (-1)^{k-l} \qquad \text{for} \quad \begin{cases} l \in \{0, \ldots, j-1\} \\ m \in \{0, \ldots, j-1-l\} \\ n \in \{m+l+1, \ldots, j\} \end{cases} \tag{4.6}$$

The simplest of these conditions are those for which $n = m + l + 1$, i.e., those for which $m = n - l - 1$, because they are the only conditions that involve sums over only two values of k:

$$\begin{aligned} 0 &= \sum_{k=l}^{l+1} A_{n-k,k} \binom{k}{l} \binom{n-k}{n-l-1} (-1)^{k-l} \\ &= A_{n-l,l} \binom{n-l}{n-l-1} - A_{n-l-1,l+1} \binom{l+1}{l} \qquad \text{for} \quad \begin{cases} n \in \{1, \ldots, j\} \\ l \in \{0, \ldots, n-1\} \end{cases} \end{aligned}$$

or, equivalently,

$$A_{n-k+1,k-1} \binom{n-k+1}{n-k} = A_{n-k,k} \binom{k}{k-1} \qquad \text{for} \quad \begin{cases} n \in \{1, \ldots, j\} \\ k \in \{1, \ldots, n\} \end{cases}$$

or, equivalently,

$$A_{n-k,k} = \frac{(n-k+1)!(k-1)!}{(n-k)!k!} A_{n-k+1,k-1} \qquad \text{for} \quad \begin{cases} n \in \{1, \ldots, j\} \\ k \in \{1, \ldots, n\} \end{cases}$$

These conditions relate pairs of coefficients $A_{n-k,k}$ having the same values of n and successive values of k. For fixed $n \in \{1, 2, \ldots, j\}$, they may be used recursively to show that $A_{n-k,k}$ must equal $A_{n,0}$ multiplied by the binomial coefficient $\binom{n}{k} \equiv n!/[k!(n-k)!]$:

$$\begin{aligned} A_{n-k,k} &= \left[\frac{(n-k+1)!\,(k-1)!}{(n-k)!\,k!}\right] \left[\frac{(n-k+2)!\,(k-2)!}{(n-k+1)!\,(k-1)!}\right] \\ &\quad \cdots \left[\frac{(n-1)!\,1!}{(n-2)!\,2!}\right] \left[\frac{n!\,0!}{(n-1)!\,1!}\right] A_{n,0} \\ &= \frac{n!\,0!}{(n-k)!\,k!} A_{n,0} = \binom{n}{k} A_{n,0} \qquad \text{for} \quad \begin{cases} n \in \{1, \ldots, j\} \\ k \in \{1, \ldots, n\} \end{cases} \end{aligned} \tag{4.7}$$

Substituting this result into the full set of conditions (4.6), using the identity

$$\binom{n}{k}\binom{k}{l}\binom{n-k}{m} = \binom{n}{m}\binom{n-m}{l}\binom{n-m-l}{k-l}$$

and then using the binomial formula gives

$$\begin{aligned}
0 &= A_{n,0} \sum_{k=l}^{n-m} \binom{n}{k}\binom{k}{l}\binom{n-k}{m}(-1)^{k-l} \\
&= A_{n,0} \binom{n}{m}\binom{n-m}{l} \sum_{k=l}^{n-m} \binom{n-m-l}{k-l}(-1)^{k-l} \\
&= A_{n,0} \binom{n}{m}\binom{n-m}{l} \sum_{k-l=0}^{n-m-l} \binom{n-m-l}{k} 1^{n-m-k}(-1)^{k-l} \\
&= A_{n,0} \binom{n}{m}\binom{n-m}{l} (1-1)^n \\
&= A_{n,0} \binom{n}{m}\binom{n-m}{l} 0^n \qquad \text{for} \quad \begin{cases} l \in \{0, \ldots, j-1\} \\ m \in \{0, \ldots, j-1-l\} \\ n \in \{m+l+1, \ldots, j\} \end{cases}
\end{aligned}$$

which shows that the remaining conditions are automatically satisfied by (4.7) without placing any further conditions on the coefficients $A_{n,0}$. The coefficients $A_{n,0}$, for $n \in \{1, \ldots, j\}$, and also the coefficient $A_{0,0}$ remain completely arbitrary.

We conclude that a linear combination of dyadic products $|z_R^-\rangle^{(l)\,(m)}\langle^- z_R|$ decays according to the pure exponential $e^{-\Gamma t}$ if and only if it is of the form

$$\sum_{n=0}^{j} A_{n,0} \sum_{k=0}^{n} \binom{n}{k} |z_R^-\rangle^{(k)\,(n-k)}\langle^- z_R| \tag{4.8}$$

with arbitrary coefficients $A_{n,0}$.

Now, coming back to the argument at the beginning of this section: We have shown that for arbitrary j, the operator $W^{\triangledown}_{(j)}$ depends only on the choice of the coefficients $A_{n,0}$. Since j was chosen arbitrarily, we can also take an operator $W^{\triangledown}$ of dimension $2j$ such that it contains at least all the terms belonging to the operator $W^{\square}_{(j)}$ of (4.1) and some additional terms which we set to zero by the choice of the coefficients $A_{hk} = 0$ for $h, k > j$,

$$\begin{aligned}
W^{\square}_{(j)} &= W^{\triangledown}_{(2j)} \qquad \text{with} \quad A_{hk} = 0 \quad \text{for } k > j \text{ or } h > j \\
&= \sum_{k=0}^{2j} \sum_{h=0}^{2j-k} A_{hk} |z_R^-\rangle^{(k)\,(h)}\langle^- z_R| \qquad \text{with} \quad A_{hk} = 0 \text{ for } k > j \text{ or } h > j
\end{aligned}$$

(4.9)

If for $n > j$ the $A_{n,0} = 0$, then we know that, according to (4.7), all the other terms $A_{n-k,k}$ are zero. Since they make up the operator $W^{\Delta}_{(j)}$ of (4.5), we conclude that the coefficients of $W^{\Delta}_{(j)}$ are zero, as demanded above.

Comparing (4.8) with arbitrary coefficients $A_{n,0}$ to the Gamow state operator (3.11) suggested by the pole term, one sees that the structure is the same with $j = r - 1$ and

$$A_{n,0} = 2\pi\Gamma\binom{r}{n+1}(-i)^n \frac{\Gamma^n}{n!} \tag{4.10}$$

5. CONCLUSION

Gamow vectors that can describe resonances and decaying states from first-order poles of the S-matrix have been known for two decades. In this paper, we discussed their generalization to Gamow vectors describing resonances from higher order S-matrix poles. This led to a set of r higher-order Gamow vectors associated with a pole of multiplicity r which are Jordan vectors to a self-adjoint Hamiltonian with complex eigenvalue $E_R - i\Gamma/2$. They are basis elements of a generalized eigenvector expansion, which suggests the form of a microphysical state associated with this higher order resonance pole. This microphysical state is a mixture of nonreducible components, and in spite of the fact that the higher order Gamow vectors have an additional polynomial time dependence, this microphysical state obeys a purely exponential decay law. We showed that this state operator has the same structure as the most general form constructed from decaying higher-order Gamow vectors that leads to an exponential decay law. It has been shown that Jordan blocks arise naturally from higher order S-matrix poles and represent a self-adjoint Hamiltonian by a complex matrix in a finite-dimensional subspace contained in the rigged Hilbert space. Although higher order S-matrix poles are not excluded theoretically, there has been very little experimental evidence for their existence, because they were always believed to have polynomial time dependence. Our results suggest that the empirical objection to the existence of higher order poles of the S-matrix does not rule out the possibility of exponentially decaying states constructed from higher order Gamow vectors.

ACKNOWLEDGMENTS

I am very happy to have been invited to Peyresq, and I wish to express my sincere gratitude to Mario Castagnino and Edgard and Diane Gunzig, who organized the meeting. A special thank you goes to Mady Smets for her extraordinary hospitality and that of her fine staff.

A little anecdote goes with this paper: The fourth section of this paper evolved as a reply to a question from one of the members of the audience during my presentation. I would like to express my gratitude to this person in cognito.

Finally, I would like to acknowledge the members of my research group, A. Bohm, M. Gadella, G. Pronko, M. Loewe, P. Patuleanu, and S. Wickramasekara, whose gracious help made this work possible and enjoyable.

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